

Solution of the Scalar Coulomb Bethe-Salpeter Equation

John H. Connell*

Springfield Technical Community College, Springfield, Massachusetts 01105

(February 1, 2008)

Abstract

A relativistic two-body wave equation, local in configuration space, is derived from the Bethe-Salpeter equation for two scalar particles bound by a scalar Coulomb interaction. The two-body bound-state wave equation is solved analytically, giving a two-body Bohr-Sommerfeld formula whose energies agree with the Bethe-Salpeter equation to order α^4 for all quantum states. From the Bohr-Sommerfeld formula, along with the expectation values of two remaining small corrections, the energy levels of the scalar Coulomb Bethe-Salpeter equation are worked out to order α^6 for all states.

Typeset using REVTeX

*connell@stcc.mass.edu

I. INTRODUCTION

For two-body atomic systems such as the hydrogen atom and positronium, it would be a great advantage to have the energy levels to order α^4 given by a relativistic two-body wave equation local in configuration space and soluble analytically. We would like to have a two-body counterpart to the one-body Coulomb Dirac equation [1] with its exact solution as found by Gordon and by Darwin in 1928.

This is a very old problem. In this paper we solve it for a system that is analogous to real atoms but simpler: two scalar particles of masses m and M , bound by a scalar Coulomb potential $-\alpha/r$.

The approach is to start from the Bethe-Salpeter equation [1] for the system. A Bethe-Salpeter equation is used because in QED it is the standard two-body bound-state equation, known to be true, and in which higher-order corrections are well understood. In QED the binding interaction in the Coulomb gauge kernel is $-\alpha/r$.

In the present paper we will treat the scalar Coulomb system's Bethe-Salpeter equation as the true equation. We will require that the energy levels of any reduction of the Bethe-Salpeter equation agree with the levels of the original Bethe-Salpeter equation.

We reduce the Bethe-Salpeter equation to a two-body relativistic bound-state wave equation which is *local* in configuration space. The locality of this reduction is in contrast to the Salpeter reduction [1] which has been the standard for the last half century. The Salpeter reduction contains operators such as $\sqrt{-\nabla^2 + m^2}$ which are non-local in configuration space and difficult to calculate with.

The relativistic two-body Coulomb wave equation has an analytic solution. The energy levels are expressed by a two-body Bohr-Sommerfeld formula.

The Bohr-Sommerfeld formula predicts the correct energy levels to order α^4 . They agree to order α^4 with the Bethe-Salpeter energy levels.

The formalism also allows calculation of the Bethe-Salpeter energy levels to order α^5 and α^6 . Because the levels up to first relativistic order, α^4 , are given by an exact solution,

only first-order perturbation theory is needed for the level calculations up to order α^6 . This is in contrast to the standard Salpeter reduction, in which the first relativistic order (α^4) already needs first-order perturbation theory, while higher orders would need second-order perturbation theory. In the present paper, explicit expressions for the Bethe-Salpeter equation's energy levels up to order α^6 are given for every quantum state.

In Section II of this paper the Bethe-Salpeter equation for the scalar system is written down, its Salpeter reduction is reviewed, and its energy levels to first relativistic order (α^4) are calculated from standard perturbation theory for later comparison.

In Section III the local relativistic two-body wave equation is derived from the Bethe-Salpeter equation. It is proven that the wave equation's first-order relativistic corrections are exactly the same as those of the originating Bethe-Salpeter equation for any static scalar interaction, so that to calculate relativistic corrections in first-order perturbation theory it is just as accurate to use the relativistic two-body wave equation as the Salpeter reduction.

In Section IV, the relativistic two-body Coulomb wave equation is solved analytically and a two-body Bohr-Sommerfeld formula for the energy levels is derived which does predict the energy levels of the Bethe-Salpeter equation correctly to order α^4 .

In Section V the formalism is used to calculate the Bethe-Salpeter energy levels up to order α^6 . One set of corrections is given by a simple expansion of the Bohr-Sommerfeld formula to order α^6 . The other two correction terms are calculated in first-order perturbation theory.

To our knowledge the work here is the first analytic solution of a relativistic reduction of a Coulomb Bethe-Salpeter equation accurate to order α^4 . It may also contain the first accurate two-body Bohr-Sommerfeld formula. We have also demonstrated that corrections to the energy levels of a Coulomb Bethe-Salpeter equation up to order α^6 can be calculated in first-order perturbation theory, once the solution is found analytically to order α^4 .

It is hoped that techniques like these may one day lead to similar results for real atomic systems such as positronium and the hydrogen atom.

II. SCALAR BETHE-SALPETER EQUATION AND SALPETER REDUCTION

A. Bethe-Salpeter Equation

The notation will be the following. Let the masses of the bound particles be m, M . Let the mass of the bound state be denoted by E , while defining the bound-state wave number below threshold as β , as well as the particles' individual CM bound-state energies as t and T , in the following way:

$$E = \sqrt{m^2 - \beta^2} + \sqrt{M^2 - \beta^2} \quad (1)$$

$$t \equiv \sqrt{m^2 - \beta^2} = \frac{E^2 + (m^2 - M^2)}{2E}, \quad (2)$$

$$T \equiv \sqrt{M^2 - \beta^2} = \frac{E^2 + (M^2 - m^2)}{2E} \quad (3)$$

The CM energy-momenta of the particles are written $(\mathbf{p}, t + p^0)$, $(-\mathbf{p}, T - p^0)$.

The Bethe-Salpeter equation for the bound-state vertex function Γ will be

$$\Gamma = I * S \Gamma \quad (4)$$

with

$$S(\mathbf{p}, p^0) \equiv - \frac{2m}{\mathbf{p}^2 + m^2 - i\epsilon - (t + p^0)^2} \frac{2M}{\mathbf{p}^2 + M^2 - i\epsilon - (T - p^0)^2} \quad (5)$$

The scalar interaction kernel is written $4mMI(\mathbf{k}^2)$, or $4mMI(r)$ in configuration space. The star denotes an integration over 4-momentum with a factor $1/i$ present. The factors $2m, 2M$ have been moved into the numerators of the propagators of the scalar constituent particles in order to make the dimensions of the scalar functions the same as in spin- $\frac{1}{2}$ systems.

B. Salpeter Reduction

Defining $\phi(\mathbf{p}) = \int dp^0 S(\mathbf{p}, p^0) \Gamma(\mathbf{p}, p^0) / 2\pi i$, equation (4) when multiplied by S and integrated over p^0 gives the non-local Salpeter equation equivalent to (4):

$$\left[\sqrt{\mathbf{p}^2 + m^2} + \sqrt{\mathbf{p}^2 + M^2} + Z(\mathbf{p})I(r) \right] \phi(\mathbf{r}) = E\phi(\mathbf{r}) \quad (6)$$

in which $Z(\mathbf{p})$ is a correction operator on the interaction $I(r)$:

$$Z(\mathbf{p}) = \frac{2mM \left[\sqrt{\mathbf{p}^2 + m^2} + \sqrt{\mathbf{p}^2 + M^2} \right]}{\sqrt{\mathbf{p}^2 + m^2} \sqrt{\mathbf{p}^2 + M^2} \left[\sqrt{\mathbf{p}^2 + m^2} + \sqrt{\mathbf{p}^2 + M^2} + E \right]} \quad (7)$$

C. First-order Relativistic Correction to Energy Levels

To calculate the first-order relativistic corrections to the Schrödinger energy levels predicted by equation (6) the non-relativistic Schrödinger equation is as usual defined to be

$$\left[m + M + \frac{\mathbf{p}^2}{2\mu} + I(r) \right] \phi_0(\mathbf{r}) = E_0 \phi_0(\mathbf{r}) \quad (8)$$

in which μ is the reduced mass. The non-relativistic bound-state wave number β_0 will be defined by

$$E_0 = m + M - \beta_0^2/2\mu \quad (9)$$

Then the Salpeter reduction (6) is expanded in powers of \mathbf{p}^2 and β_0^2 in the standard way, giving the correction $\Delta E = E - E_0$:

$$\Delta E = -\frac{\beta_0^4}{8} \left(\frac{1}{m^3} + \frac{1}{M^3} \right) + \frac{\beta_0^2}{2mM} \langle \phi_0 | I | \phi_0 \rangle + \frac{1}{2\mu} \left(\frac{\mu^2}{m^2} + \frac{\mu^2}{M^2} \right) \langle \phi_0 | I^2 | \phi_0 \rangle \quad (10)$$

This constitutes the first-order relativistic correction for the energy level of the original Bethe-Salpeter equation (4).

In the scalar Coulomb case

$$I(r) = -\frac{\alpha}{r} \quad (11)$$

equation (10) shows that the energy levels of the scalar Coulomb Bethe-Salpeter equation have this correction to order α^4 :

$$\Delta E_{\text{Coul}}^{(4)} = \frac{\alpha^4 \mu}{N^3(2L+1)} \left(\frac{\mu^2}{m^2} + \frac{\mu^2}{M^2} \right) - \frac{\alpha^4 \mu}{8N^4} \left(1 + \frac{\mu^2}{mM} \right) \quad (12)$$

where L is the angular momentum and N is the Bohr quantum number.

III. RELATIVISTIC WAVE EQUATION

A. Derivation

We return to the original Bethe-Salpeter equation (4) and reduce it to a three-dimensional form in a manner different from that of Salpeter.

Because S peaks sharply at $p^0 = 0$ when \mathbf{p}^2 is not too large, it can be approximated [2–4] by

$$S_0(\mathbf{p}, p^0) \equiv -\frac{4mM}{2E} \frac{2\pi i \delta(p^0)}{(\mathbf{p}^2 + \beta^2)} \quad (13)$$

The error of the approximation is defined by

$$R \equiv S - S_0 \quad (14)$$

The Blankenbecler-Sugar [5] correction interaction U is defined by the equation

$$U = I + I * R U \quad (15)$$

Then from the Bethe-Salpeter equation (4) it is easy to deduce the equation $\Gamma = U * S_0 \Gamma$. Next, defining $\phi(\mathbf{p})$ by $S_0 \Gamma = 2\pi i \delta(p^0) \phi(\mathbf{p})$, it follows that, with the notation $U = U(p; q)$,

$$(\mathbf{p}^2 + \beta^2) \phi(\mathbf{p}) = -\frac{4mM}{2E} \int \frac{d\mathbf{q}}{(2\pi)^3} U(\mathbf{p}, 0; \mathbf{q}, 0) \phi(\mathbf{q}) \quad (16)$$

Equation (16) is exact, in terms of the original Bethe-Salpeter equation (4).

For a static kernel $I(\mathbf{k}^2)$, equation (15) for U simplifies, because in its solution by iteration the integration over the relative-energy variable only acts on R . After that integration, equation (15) becomes

$$U(\mathbf{p}, 0; \mathbf{q}, 0) = I[(\mathbf{p} - \mathbf{q})^2] + \int \frac{d\mathbf{l}}{(2\pi)^3} I[(\mathbf{p} - \mathbf{l})^2] R(\mathbf{l}^2) U(\mathbf{l}, 0; \mathbf{q}, 0) \quad (17)$$

in which

$$R(\mathbf{l}^2) = \frac{mM}{E} \left\{ \frac{1}{\sqrt{\mathbf{l}^2 + m^2} (\sqrt{\mathbf{l}^2 + m^2} + t)} + \frac{1}{\sqrt{\mathbf{l}^2 + M^2} (\sqrt{\mathbf{l}^2 + M^2} + T)} \right\} \quad (18)$$

where the energy-dependent constants t and T were defined in (2) and (3).

B. Local Equation

Equation (17) for U leads immediately to a local wave equation accurate to first relativistic order.

Because in many systems the momentum is small compared to the constituent masses, it is a reasonable starting point to approximate (18) by its value at $\mathbf{l}^2 = 0$:

$$R_E \equiv R(0) = \frac{mM}{E} \left[\frac{1}{m(m+t)} + \frac{1}{M(M+T)} \right] \quad (19)$$

Then in configuration space equation (17) can be iterated once to give a local approximation to U :

$$U_E(r) = I(r) + R_E I^2(r) \quad (20)$$

Defining in addition the quantity

$$Z_E \equiv \frac{m+M}{E} \quad (21)$$

equation (16) gives

$$\left\{ \mathbf{p}^2 + 2\mu Z_E \left[I(r) + R_E I^2(r) \right] \right\} \phi(\mathbf{r}) = -\beta^2 \phi(\mathbf{r}) \quad (22)$$

which is a two-body wave equation local in configuration space. Equation (22) contains energy-dependent constants, instead of the non-local operators of the Salpeter reduction (6). The eigenvalue β^2 determines the energy E through equation (1).

C. Equivalence Proof

Equation (22) can be expanded around the Schrödinger equation (8) to calculate the energy corrections given by (22) to the first relativistic order.

Expression (1) for the energy must be expanded to fourth order in β_0 , so that

$$\Delta E = -\frac{\beta_0^4}{8} \left(\frac{1}{m^3} + \frac{1}{M^3} \right) - \frac{1}{2\mu} \Delta \beta^2 \quad (23)$$

In evaluating $\Delta\beta^2 \equiv \beta^2 - \beta_0^2$ from (22), it is only necessary to express R_E to zero order in β_0 , as $(\mu^2/m^2 + \mu^2/M^2)/2\mu$. In addition Z_E need only be expanded to second order in β_0 , as $1 + \beta_0^2/2mM$; in the I^2 term Z_E can be taken to be 1.

From these values (22) immediately gives the first-order correction

$$\Delta\beta^2 = -\frac{\beta_0^2}{m+M}\langle I \rangle - \left(\frac{\mu^2}{m^2} + \frac{\mu^2}{M^2} \right) \langle I^2 \rangle \quad (24)$$

Substitution of (24) into (23) shows that the expression for ΔE is the same as the expression (10) derived by means of the Salpeter reduction.

This constitutes a proof that the local relativistic two-body wave equation (22) correctly gives the first-order relativistic energy corrections to the Bethe-Salpeter equation (4). A similar proof for spin- $\frac{1}{2}$ particles was derived some time ago [6].

D. Infinite-Mass Limit

In the limit $M \rightarrow \infty$, equation (22) becomes

$$\left\{ \mathbf{p}^2 + 2mI(r) + \frac{2m}{m+t}I^2(r) \right\} \phi(\mathbf{r}) = -(m^2 - t^2) \phi(\mathbf{r}) \quad (25)$$

where t , defined in (2), is the bound-state energy of the lighter particle.

To first relativistic order the term $2mI^2/(m+t)$ can be replaced by I^2 . Then (25) becomes

$$\left\{ \mathbf{p}^2 + [m + I(r)]^2 \right\} \phi(\mathbf{r}) = t^2 \phi(\mathbf{r}) \quad (26)$$

which is the customary Klein-Gordon equation for a scalar particle in a fixed field.

IV. SCALAR COULOMB SOLUTION

Specialising to the scalar Coulomb interaction (11), the Blankenbecler-Sugar correction potential given by (17) becomes to second order

$$U_{\text{Coul}}(r) = -\frac{\alpha}{r} + R_E \frac{\alpha^2}{r^2} \quad (27)$$

and the relativistic two-body wave equation (22) becomes

$$\left\{ \mathbf{p}^2 + 2\mu Z_E \left[-\frac{\alpha}{r} + R_E \frac{\alpha^2}{r^2} \right] \right\} \phi = -\beta^2 \phi \quad (28)$$

The radial reduction of equation (28) has the same singularities as that of the Coulomb Schrödinger equation, and can be solved the same way. For angular momentum L the radial component can be expanded for radial quantum numbers $n = 0, 1, 2, \dots$ as

$$e^{-\beta r} (\beta r)^\epsilon \sum_{i=0}^n a_i (\beta r)^i \quad (29)$$

which as usual gives a recurrence relation between the coefficients a_i . The existence of a_0 , and the termination of the series, require the conditions

$$\epsilon = \sqrt{\left(L + \frac{1}{2}\right)^2 + \alpha^2 2\mu Z_E R_E} - \frac{1}{2} \quad (30)$$

and

$$\beta = \frac{Z_E \alpha \mu}{n + 1 + \epsilon} \quad (31)$$

Then from (31) and expression (1) for the energy E , it is easy to deduce the Bohr-Sommerfeld formula

$$E = \sqrt{m^2 + M^2 + 2mM} \sqrt{1 - \frac{\alpha^2}{(n + 1 + \epsilon)^2}} \quad (32)$$

Expanding in the usual way, with Z_E and R_E inside ϵ only needed to zero order, it is found that the Bohr-Sommerfeld expression (32) does predict the correct α^4 correction (12) to the energy levels of the scalar Coulomb Bethe-Salpeter equation, as expected.

V. ENERGY LEVELS TO ORDER α^6

A. Introduction

Up to this point it has only been shown that the reduction of the Bethe-Salpeter equation to a relativistic two-body wave equation as described in the previous section reproduces the results of the standard Salpeter reduction to order α^4 , albeit through an analytic solution instead of perturbation theory. One way of investigating whether the present reduction may be more useful than the method of Salpeter would be to find out whether higher-order terms in the energy levels can be calculated more easily than the Salpeter reduction allows. In the present section we will calculate the energy levels of the Bethe-Salpeter equation (4) analytically to order α^6 for all quantum states, using only first-order perturbation theory.

With the definition

$$\Delta R(\mathbf{l}^2) \equiv R(\mathbf{l}^2) - R_E \quad (33)$$

the expansion of equation (17) for U becomes (with I standing for $-\alpha/r$)

$$U = U_{\text{Coul}} + I \Delta R I + IRIRI + IRIRIRI + \dots \quad (34)$$

We will evaluate the contributions to the energy levels up to order α^6 due to:

- U_{Coul} , whose levels are contained in the Bohr-Sommerfeld formula (32);
- $I \Delta R I$, which will have an α^5 contribution for $L = 0$, as well as α^6 contributions;
- the order- α^6 contribution of the rest of the series,

$$U_6 = IRIRI + IRIRIRI + \dots \quad (35)$$

B. Bohr-Sommerfeld Formula

The Bohr-Sommerfeld expression (32) can be expanded further to get the sixth-order contribution to the energy of the Blankenbecler-Sugar potential U_{Coul} , equation (27). Defining the dimensionless constant

$$c \equiv \frac{\mu^2}{m^2} + \frac{\mu^2}{M^2} \quad (36)$$

the sixth-order energy contribution is

$$\begin{aligned} \Delta E_{\text{Bohr-Somm}}^{(6)} = \alpha^6 \mu \left\{ \frac{4c - 7 - c^2}{64N^6} + \right. \\ \left. + \frac{12c - 5c^2 - 1}{8N^5(2L + 1)} - \frac{3c^2}{2N^4(2L + 1)^2} - \frac{c^2}{N^3(2L + 1)^3} \right\} \end{aligned} \quad (37)$$

We recall that L is the angular momentum and N is the Bohr quantum number. To obtain this expression the constants Z_E , R_E and ϵ are expanded to the order in α needed.

C. Correction $I \Delta R I$

Following the steps outlined in Subsection A above, we calculate the α^5 and α^6 energy corrections due to the term $I \Delta R I$ in (34), where ΔR is defined in (33). The correction is

$$\Delta E_{\Delta R}^{(6)} = \langle \phi | I \Delta R I | \phi \rangle \quad (38)$$

Up to order α^6 it is sufficient to use non-relativistic wavefunctions. Use of the Schrödinger equation (8) then gives

$$\Delta E_{\Delta R}^{(6)} = \frac{1}{(2\mu)^2} \int \frac{d\mathbf{p}}{(2\pi)^3} \phi_0^*(\mathbf{p}) \left[\mathbf{p}^2 + \beta_0^2 \right]^2 \Delta R(\mathbf{p}^2) \phi_0(\mathbf{p}) \quad (39)$$

To evaluate the integral (39) for every quantum state, it is convenient to change coordinates from \mathbf{p} -space to the surface of Schwinger's unit sphere [7] in Cutkosky's 4-space [4]. The transformation is (with $i = 1, 2, 3$)

$$\xi_i = \frac{2\beta \mathbf{p}_i}{\mathbf{p}^2 + \beta^2}, \quad \xi_4 = \frac{\mathbf{p}^2 - \beta^2}{\mathbf{p}^2 + \beta^2} \quad (40)$$

(Henceforth β is written for β_0 .) The polar coördinates on the unit sphere are conventionally denoted by $(\theta_1, \theta_2, \phi)$ [8] where θ_2, ϕ are the usual angles of 3-space, and

$$x \equiv \cos \theta_1 = \xi_4 \quad (41)$$

The element of surface area is [8]

$$d\Omega = (\sin \theta_1)^2 (\sin \theta_2) d\theta_1 d\theta_2 d\phi \quad (42)$$

and $d\Omega$ and $d\mathbf{p}$ are related by [7]

$$d\Omega = \left[\frac{2\beta}{\mathbf{p}^2 + \beta^2} \right]^3 d\mathbf{p} \quad (43)$$

Surface harmonics on Schwinger's unit sphere are denoted by Y_{NLM} , where $N - 1 \geq L \geq |M| \geq 0$. The quantum numbers L, M have their usual meaning in 3-space, and $N = 1, 2, 3, \dots$ is the Bohr quantum number. As usual they are related by $N = n + L + 1$ where $n = 0, 1, 2, \dots$ is the radial quantum number in \mathbf{p} -space.

The standard representation of the normalised surface harmonics is [8]

$$Y_{NLM}(\theta_1, \theta_2, \phi) = 2^{L+1} \Gamma(L+1) \sqrt{\frac{n! N}{2\pi \Gamma(N+L+1)}} (1-x^2)^{\frac{L}{2}} C_n^{L+1}(x) Y_{LM}(\theta_2, \phi) \quad (44)$$

in which $Y_{LM}(\theta_2, \phi)$ is normalised on the surface of the unit sphere in 3 dimensions, and the coefficient is determined by the orthonormality relation [8]

$$\int_{-1}^{+1} (1-x^2)^{L+\frac{1}{2}} C_m^{L+1}(x) C_n^{L+1}(x) dx = \delta_{m,n} \frac{\pi \Gamma(N+L+1)}{n! N [\Gamma(L+1)]^2 2^{2L+1}} \quad (45)$$

The momentum-space eigenfunctions $\phi_0(\mathbf{p})$ are [7] proportional to $(1-x)^2 Y_{NLM}$. With the normalisation requirement

$$\int \frac{d\mathbf{p}}{(2\pi)^3} |\phi_0(\mathbf{p})|^2 = 1 \quad (46)$$

we find that

$$\phi_0(\mathbf{p}) = \left[\frac{2\pi}{\beta} \right]^{\frac{3}{2}} (1-x)^2 Y_{NLM}(\theta_1, \theta_2, \phi) \quad (47)$$

Then substitution of equation (47) into (39) gives

$$\Delta E_{\Delta R}^{(6)} = \frac{2^{2L+1} n! N [\Gamma(L+1)]^2 \beta^4}{\pi \Gamma(N+L+1) \mu^2} I_{L,n} \quad (48)$$

in which

$$I_{L,n} \equiv \int_{-1}^{+1} dx (1-x^2)^{L+\frac{1}{2}} [C_n^{L+1}(x)]^2 \frac{\Delta R(\mathbf{p}^2)}{1-x} \quad (49)$$

Although $\Delta R(0) = 0$, it is not possible to expand $\Delta R(\mathbf{p}^2)$ to first order as a Taylor series in \mathbf{p}^2 , because the integrals in (39) and (49) would diverge when $L = 0$. In order to evaluate $I_{L,n}$ for all states including $L = 0$ it is necessary to express $\Delta R(\mathbf{p}^2)$ exactly.

With successive definitions, referring to equations (40) and (41):

$$a \equiv \frac{2\beta^2}{t^2}, \quad A \equiv \frac{2\beta^2}{T^2} \quad (50)$$

$$D(y, a, t, m) \equiv a \left\{ \frac{5\sqrt{y} + 4\sqrt{y+a}}{3(\sqrt{y} + \sqrt{y+a})^2} + \frac{t^2(m+2t)}{2m(m+t)^2} \sqrt{y} \right\} \quad (51)$$

$$F(x) \equiv -\frac{mM}{E} \left\{ \frac{1}{t^2} D(1-x, a, t, m) + \frac{1}{T^2} D(1-x, A, T, M) \right\} \quad (52)$$

it can be shown that

$$\frac{\Delta R(\mathbf{p}^2)}{\sqrt{1-x}} = \frac{d}{dx} F(x) \quad (53)$$

Expression (49) can now be integrated by parts. To the required order, it becomes

$$I_{L,n} = -\alpha \delta_{L,0} \frac{8N}{3\mu} \left[\frac{\mu^3}{m^3} + \frac{\mu^3}{M^3} \right] - J_{L,n} \quad (54)$$

in which

$$J_{L,n} = \int_{-1}^{+1} dx F(x) \frac{d}{dx} \left\{ \sqrt{1+x} (1-x^2)^L [C_n^{L+1}(x)]^2 \right\} \quad (55)$$

Note that the $\delta_{L,0}$ term is of order α , not α^2 . It will give a contribution of order $\alpha^5 \mu$ to the energy. In atomic physics the Coulomb potential gives an α^5 contribution when $L = 0$. In the present model and formalism that term appears in the ΔR correction.

The remaining integral (55) stays convergent as $a, A \rightarrow 0$ inside the square roots in the functions D contained in $F(x)$. To the required order that limit may be taken, and (55) becomes to lowest order

$$J_{L,n} = -\alpha^2 \frac{1}{\mu N^2} \left[\frac{\mu^4}{m^4} + \frac{\mu^4}{M^4} \right] (K_{L,n}^{(1)} + K_{L,n}^{(2)}) \quad (56)$$

in which

$$K_{L,n}^{(1)} = \frac{3}{4} \int_{-1}^{+1} dx \sqrt{1-x} \frac{d}{dx} \left\{ \sqrt{1+x} (1-x^2)^L [C_n^{L+1}(x)]^2 \right\} \quad (57)$$

$$K_{L,n}^{(2)} = \frac{3}{2} \int_{-1}^{+1} dx \frac{1}{\sqrt{1-x}} \frac{d}{dx} \left\{ \sqrt{1+x} (1-x^2)^L [C_n^{L+1}(x)]^2 \right\} \quad (58)$$

The expression for $K_{L,n}^{(1)}$ can be integrated by parts for all L . The expression for $K_{L,n}^{(2)}$ can be integrated directly by parts for $L \geq 1$, while for $L = 0$ the integration by parts may be done by subtracting $[C_n^{L+1}(1)]^2$ from $[C_n^{L+1}(x)]^2$ in the integrand beforehand. The subtracted part is calculated separately and added back after the integral is evaluated. With that understanding, we have

$$K_{L,n}^{(1)} = +\frac{3}{8} \int_{-1}^{+1} \frac{(1-x^2)^{L+\frac{1}{2}} [C_n^{L+1}(x)]^2}{1-x} dx \quad (59)$$

$$K_{L,n}^{(2)} = -\frac{3}{4} \int_{-1}^{+1} \frac{(1-x^2)^{L+\frac{1}{2}} [C_n^{L+1}(x)]^2}{(1-x)^2} dx \quad (60)$$

To evaluate these $K_{L,n}$ integrals it has been necessary to derive the following equation [9] (in which $m \leq n$):

$$\begin{aligned} & \int_{-1}^{+1} dx \frac{(1-x^2)^{L+\frac{1}{2}} C_m^{L+1}(x) C_n^{L+1}(x)}{z-x} \\ &= \frac{\sqrt{\pi} \Gamma(N+L+1)}{2^{N+L} \Gamma(L+1) \Gamma(N+\frac{1}{2})} C_m^{L+1}(z) \int_{-1}^{+1} dt \frac{(1-t^2)^{n+L+\frac{1}{2}}}{(z-t)^{n+1}} \end{aligned} \quad (61)$$

With $m = n$ the limit $z \rightarrow 1$ gives

$$K_{L,n}^{(1)} = \frac{3}{8} \frac{\sqrt{\pi} \Gamma(L+\frac{1}{2}) \Gamma(N+L+1)}{\Gamma(n+1) \Gamma(L+1) \Gamma(2L+2)} \quad (62)$$

To evaluate (60) one can differentiate (61) with respect to z and take the limit $z \rightarrow 1$. For $L = 0$ the subtracted integral is calculated before taking the limit $z \rightarrow 1$; the value is then finite. For all L , including $L = 0$, the result is

$$K_{L,n}^{(2)} = \frac{3}{4} \frac{\sqrt{\pi} \Gamma(L + \frac{1}{2}) \Gamma(N + L + 1)}{\Gamma(n + 1) \Gamma(L + 1) \Gamma(2L + 2)} \frac{1 - 4N^2}{(2L - 1)(2L + 3)} \quad (63)$$

These evaluations are carried out using standard representations and properties of the beta function and the hypergeometric function.

Working back from these results to the original expression (48) for the energy correction, we finally have

$$\begin{aligned} \Delta E_{\Delta R}^{(6)} = & -\alpha^5 \mu \delta_{L,0} \frac{16}{3\pi N^3} \left[\frac{\mu^3}{m^3} + \frac{\mu^3}{M^3} \right] + \\ & + \alpha^6 \mu \frac{3}{2N^5(2L+1)} \left[\frac{\mu^4}{m^4} + \frac{\mu^4}{M^4} \right] \left[\frac{1}{2} + \frac{1 - 4N^2}{(2L+3)(2L-1)} \right] \end{aligned} \quad (64)$$

The reader is reminded that this is the energy correction to order α^6 due to the correction $\Delta R = R(\mathbf{p}^2) - R_E$, and that its calculation used first-order perturbation theory only.

D. Correction $IRIRI + IRIRIRI + \dots$

Finally we address the rest of the series for U , given by (35).

It will be proven below that to the required order $R(\mathbf{p}^2)$ can be approximated by its lowest-order value $c/2\mu$, where the dimensionless function of the masses c is defined by (36). Then the series (35) becomes to lowest order

$$U_6^0(r) = I(c/2\mu)I(c/2\mu)I + I(c/2\mu)I(c/2\mu)I(c/2\mu)I + \dots \quad (65)$$

$$= \frac{\alpha^3(c/2\mu)^2}{r^2(r + \alpha c/2\mu)} \quad (66)$$

The energy correction up to order α^6 is given by the expectation value of (66) over non-relativistic Coulomb wavefunctions.

For $L \geq 1$ the lowest-order expectation value is just dictated by the expectation value of $1/r^3$. It is

$$\Delta E_{U6}^{(6)} \Big|_{L \geq 1} = \langle U_6(r) \rangle_{L \geq 1} = -\alpha^6 \mu \frac{c^2}{2N^3 L(L+1)(2L+1)} \quad (67)$$

For $L = 0$ a logarithm occurs. Unfortunately expectation values of quantities of the form (66) over Coulomb wavefunctions do not seem to be listed in standard references. By

expanding the polynomial in the radial wavefunctions the result can easily be found as a double sum from 0 to n . But in case this kind of integral arises later in real atomic two-body systems, at the cost of a few more lines of calculation we present an evaluation in closed form.

In terms of the variable $z = 2\beta_0 r = 2\alpha\mu r/N$, the radial wavefunction is known to be $z^L e^{-z/2} F(-n, 2L+2, z)$ up to normalisation, where F is the confluent hypergeometric function. Recalling that $L = 0$, and defining $a = \alpha^2 c/N$, we need to evaluate

$$K_a \equiv \int_0^\infty e^{-z} \frac{1}{z+a} [F(-n, 2, z)]^2 dz \quad (68)$$

While this expression has not been found in any reference, Landau and Lifshitz [10] give an evaluation of a related integral:

$$J_\nu \equiv \int_0^\infty e^{-z} z^{\nu-1} [F(-n, \gamma, z)]^2 dz \quad (69)$$

$$= \frac{\Gamma(\nu)n!}{(\gamma)_n} \sum_{i=0}^n \frac{(-n)_i (\nu - \gamma + 1)_i (\gamma - \nu)_i}{[i!]^2 (\gamma)_i} \quad (70)$$

where as usual $(\kappa)_k = \kappa(\kappa+1)\cdots(\kappa+k-1)$, $(\kappa)_0 = 1$. Here the quantity γ will be 2 since $L = 0$. The value $\nu = 3$ gives the normalisation integral $J_3 = 2$.

In K_a the parameter a is small and a $\log(1/a)$ term will dominate, followed by $O(1)$ terms. These are the terms needed to find the energy to order α^6 . Also, if ν is taken to be small in J_ν a $1/\nu$ term will dominate and the next terms will be $O(1)$. If $[F(-n, 2, z)]^2$ is written as $1 + B(z)$, in $B(z)$ the lowest power of z will be one. Then it is clear that in both K_a and J_ν , the leading contribution of the part $B(z)$ is of order 1. Furthermore, those $O(1)$ terms can be calculated by replacing $1/(z+a)$ by $1/z$ and $z^{\nu-1}$ by $1/z$ respectively. In other words, the contribution of $B(z)$ to the $O(1)$ terms of both integrals is identical. In addition, the contribution of the “1” in each integral can be calculated explicitly.

Therefore the way to evaluate the unknown integral (68) up to $O(1)$ for small a is the following. (i) From the result (70) subtract the explicitly calculated contribution to J_ν of the “1” in $[F]^2$. (ii) Evaluate the now leading $O(1)$ part of the result, neglecting terms of order ν . (iii) Explicitly calculate the contribution of the “1” to K_a to orders $\log(1/a)$ and

1, neglecting terms of order a . (iv) Add that to the result (ii), finally obtaining the $\log(1/a)$ and $O(1)$ terms of K_a as required. The result is

$$K_a = \log \frac{1}{a} - (n+2)\mathbf{C} + \frac{n(4n+1)}{2(n+1)} - (n+1)\psi(n+1) + O(a) \quad (71)$$

and hence to order α^6

$$\begin{aligned} \Delta E_{U6}^{(6)}|_{L=0} &= \langle U_6(r) \rangle_{L=0} = \\ &= \alpha^6 \mu \frac{c^2}{N^3} \left\{ \log \frac{N}{\alpha^2 c} - (N+1)\mathbf{C} + \frac{n(4n+1)}{2N} - N\psi(N) \right\} \end{aligned} \quad (72)$$

in which $N = n+1$, $\psi(z) \equiv d \log[(\Gamma(z))/dz]$ and \mathbf{C} is Euler's constant $0.5772\dots$

At the beginning of this Subsection we stated that it was sufficiently accurate to replace $R(\mathbf{p}^2)$ by $c/2\mu$ in the series (35). Now it is necessary to prove that the correction to this approximation is of order α^7 or greater. With the definition $\delta R = R(\mathbf{p}^2) - c/2\mu$, the correction to the α^6 term $IRIRI$ will be

$$I\delta RI(c/2\mu)I + I(c/2\mu)I\delta RI \quad (73)$$

We must prove that this term is of order α^7 only. Since $R_E - c/2\mu = O(\alpha^2)$, we have $\delta R = \Delta R + O(\alpha^2)$, and it is sufficient to prove that

$$I\Delta RI(c/2\mu)I + I(c/2\mu)I\Delta RI \quad (74)$$

is of order α^7 .

The calculations of Subsection C suggest that (74) should be of order α^7 , since $\langle I\Delta RI \rangle$ was of order α^5 while $\langle I(c/2\mu)I \rangle$ is of order α^4 . However, it is not quite obvious, since $\Delta R(\infty) = -R_E \approx -c/2\mu$, and therefore the rules of power-counting would permit (74) to be of order α^6 . To see whether it is or not needs an explicit calculation.

In momentum space, each term of (74) contains a factor of the form

$$\int \frac{d\mathbf{l}}{(2\pi)^3} \frac{e^2}{(\mathbf{p}-\mathbf{l})^2} \Delta R(\mathbf{l}^2) \frac{e^2}{(\mathbf{l}-\mathbf{q})^2} \quad (75)$$

(preceded or succeeded by other factors) while in the first (α^6) term of (65) the corresponding factor is

$$\int \frac{d\mathbf{l}}{(2\pi)^3} \frac{e^2}{(\mathbf{p}-\mathbf{l})^2} \frac{c}{2\mu} \frac{e^2}{(\mathbf{l}-\mathbf{q})^2} \quad (76)$$

We will exhibit counterpart terms in these two expressions and show that the one in (75) is smaller than the other by a factor $O(\alpha)$.

Again it is necessary to use the Schwinger unit sphere. With the mapping $\mathbf{p} \rightarrow \xi$ as before, as well as $\mathbf{l} \rightarrow \eta$ and $\mathbf{q} \rightarrow \zeta$, it is true that [7]

$$\frac{e^2}{(\mathbf{p}-\mathbf{l})^2} = \frac{1-\xi_4}{\beta} \frac{e^2}{(\xi-\eta)^2} \frac{1-\eta_4}{\beta} \quad (77)$$

$$\frac{e^2}{(\xi-\eta)^2} = (2\pi)^3 \alpha \sum_{NLM} \frac{1}{N} Y_{NLM}(\xi) Y_{NLM}^*(\eta) \quad (78)$$

Using the various relations given in Subsection C, expression (76) becomes for fixed L, M

$$\frac{1-\xi_4}{\beta} \sum_{N'N''} \frac{(2\pi)^3 \alpha^2}{N'N''} Y_{N'LM}(\xi) A_{L,N'N''}^{(c/2\mu)} Y_{N''LM}^*(\zeta) \frac{1-\zeta_4}{\beta} \quad (79)$$

in which

$$A_{L,N',N''}^{(c/2\mu)} = \int d\Omega_\eta Y_{N'LM}^*(\eta) \frac{\beta c}{2\mu} \frac{1}{1-\eta_4} Y_{N''LM}(\eta) \quad (80)$$

With the notations $N' = n' + L + 1$, $N'' = n'' + L + 1$, this is

$$A_{L,N',N''}^{(c/2\mu)} = \left[2^{L+1} \Gamma(L+1) \right]^2 \sqrt{\frac{n'! N'}{2\pi \Gamma(N'+L+1)}} \sqrt{\frac{n''! N''}{2\pi \Gamma(N''+L+1)}} B_{L,N',N''}^{(c/2\mu)} \quad (81)$$

where, with $y \equiv \eta_4$,

$$B_{L,N',N''}^{(c/2\mu)} = \frac{\beta c}{2\mu} \int_{-1}^{+1} dy \frac{(1-y^2)^{L+\frac{1}{2}} C_{n'}^{L+1}(y) C_{n''}^{L+1}(y)}{1-y} \quad (82)$$

Expression (82) can be evaluated with the help of equation (61). Supposing for definiteness that $n' > n''$, we easily find

$$B_{L,N',N''}^{(c/2\mu)} = \frac{\beta c}{2\mu} \frac{\sqrt{\pi} \Gamma(L+\frac{1}{2}) \Gamma(N''+L+1)}{n''! \Gamma(2L+2) \Gamma(L+1)} \quad (83)$$

Next we do the corresponding decomposition of (75). The steps are the same, with ΔR replacing $c/2\mu$. Using the representation (53) as before, we end up with

$$B_{L,N',N''}^{(\Delta R)} = -\frac{8}{3} \delta_{L,0} \frac{\beta^2}{\mu^2} N' N'' \left[\frac{\mu^3}{m^3} + \frac{\mu^3}{M^3} \right] - \beta J_{L,n',n''} \quad (84)$$

where

$$J_{L,n',n''} = \int_{-1}^{+1} dy F(y) \frac{d}{dy} \left\{ \sqrt{1+y(1-y^2)}^L C_{n'}^{L+1}(y) C_{n''}^{L+1}(y) \right\} \quad (85)$$

The expression for $J_{L,n',n''}$ is evaluated the same way that (55) was. Similarly to $J_{L,n}$, it is found to be of order α^2 .

Therefore, recalling that $\beta = O(\alpha)$, we finally have

$$B_{L,N',N''}^{(\Delta R)} = O(\alpha^2) + O(\alpha^3) \quad (86)$$

while

$$B_{L,N',N''}^{(c/2\mu)} = O(\alpha) \quad (87)$$

Thus the expression (75) is of order α smaller than (76), and so the correction term (74) is a factor α smaller than the original quantity $I(c/2\mu)I(c/2\mu)I$ which is of order α^6 . It follows that the sum (66) does correctly represent the series (35) up to and including order α^6 .

E. Summary

The complete energy corrections of order α^5 , $\alpha^6 \log(1/\alpha)$, and α^6 , to the Bethe-Salpeter equation (4) with a scalar Coulomb kernel, are given by the sum of:

- Equation (37), from the two-body Bohr-Sommerfeld formula (32);
- Equation (64), from the correction $I\Delta RI$ to U ;
- Equation (67) for $L \geq 1$, and equation (72) for $L = 0$, which came from the sum (66) of the rest of the Coulomb series for U .

These corrections were calculated for every bound state, not just for a few low-lying states. The first correction was calculated by simple algebra. The latter two were calculated from first-order perturbation theory. As far as we know, calculations like these are beyond the power of the Salpeter reduction.

VI. CONCLUSION

Advantages of the formalism shown here are that the local wave equation is derived directly from the Bethe-Salpeter equation, and that the two-body wave equation is accurate enough to predict the energy levels of the Bethe-Salpeter equation to first relativistic order correctly.

Two-body relativistic wave equations such as (28) are easier to solve than Salpeter equations such as (6), because they are local in configuration space. The non-local components of the Salpeter equation, such as the free-particle kinetic-energy operator $\sqrt{\mathbf{p}^2 + m^2} + \sqrt{\mathbf{p}^2 + M^2}$, and the correction factor Z_p shown in equation (7), are replaced in the two-body relativistic wave equation by constants, such as $E = \sqrt{m^2 - \beta^2} + \sqrt{M^2 - \beta^2}$, $Z_E = (m + M)/E$, and the correction factor R_E shown in equation (19).

In addition, when the kernel of the scalar Bethe-Salpeter equation is a scalar Coulomb potential, the relativistic two-body bound-state wave equation is soluble exactly. The resultant two-body Bohr-Sommerfeld formula not only predicts the Bethe-Salpeter energy levels to order α^4 correctly, but it also allows much of the $O(\alpha^6)$ correction to be evaluated by simple algebra. Furthermore, the remaining α^5 and α^6 corrections are calculated by first-order perturbation theory only. No second-order perturbation calculations are needed.

The results shown above suggest that it may be easier to solve Bethe-Salpeter equations, at least those whose binding interaction is a Coulomb potential, with the aid of a local wave equation rather than a Salpeter equation. It is hoped that the formalism developed here can be adapted to real two-body atoms such as the hydrogen atom and positronium.

I thank the trustees of Springfield Technical Community College for a half-time sabbatical leave which assisted the completion of this work. I am grateful to G. Adkins for extensive comments.

REFERENCES

- [1] For all historical references, see H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag/Academic, New York, 1957) [also published in *Handbuch der Physik*, Vol. **XXXV** (Springer, Berlin/Göttingen/Heidelberg, 1956)].
- [2] J. H. Connell, Thesis, University of Washington (Seattle, 1967).
- [3] I.T. Todorov, Phys. Rev. **D3**, 2351 (1971).
- [4] J.H. Connell, Phys. Rev. **D43**, 1393 (1991).
- [5] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
- [6] J.H. Connell, SLAC-PUB 5633 (1991).
- [7] J. Schwinger, J. Math. Phys. **5**, 1606 (1964).
- [8] H. Erdélyi, ed., *Higher Transcendental Functions*, Vol. II (McGraw-Hill, 1953).
- [9] I.S. Gradshteyn & I.M. Ryzhik, *Tablitsy Integralov, Summ, Ryadov i Proizvedenii* (Gos. Isd. Fiz.-Mat. Literaturny, Moscow, 1963). Our equation (61) is derived from formulae 7.312(3) and 8.712 of this reference, but we have had to insert an apparently missing factor of 2. Equation (61) is verified by the fact that its multiplication by z followed by the limit $z \rightarrow \infty$ predicts the orthonormality relation (45), and that its limit as $z \rightarrow 1$ (our result (62)) agrees with formula 7.314(1) of this reference. Formula 7.314(7), which would have served to evaluate both our $K_{L,n}$ integrals, disagrees with known special cases and is apparently in error.
- [10] L. D. Landau and E. M. Lifshitz, *Kvantovaya Mekhanika, Nerelevativistskaya Teoriya*, 2nd edition (Fizmatgiz, Moscow 1963) [Translation: *Quantum Mechanics, Non-Relativistic Theory*, 1st edition (Pergamon Press-Addison Wesley, 1958)], Appendix.